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ABSTRACT

This paper describes the Rational Number Project (RNP), teaching experiments concerned with the teaching and learning of fractions among 4th and 5th grade students. Interviews with 4th grade students who used the RNP curriculum and with students who used a traditional curriculum were conducted by RNP staff as well as classroom teachers. This paper provides teachers with examples of students' conceptual and procedural thinking to give them a picture of what it means for students to exhibit number sense with fractions. Focusing on children's thinking and the effect of the use of manipulative models on children's thinking is suggested as one way of teaching fractions. (KHR)

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Using Manipulative Models to Build Number Sense for Addition of Fractions

Kathleen Cramer

Apryl Henry

THE Rational Number Project (RNP) has reported on several long-term teaching experiments concerning the teaching and learning of fractions among fourth- and fifth-grade students (Bezuk and Cramer 1989; Post et al. 1985). A curriculum created for these teaching experiments and then revised on the basis of what was learned reflects the following four beliefs: (1) children's learning about fractions can be optimized through active involvement with multiple concrete models, (2) most children need to use concrete models over extended periods of time to develop mental images needed to think conceptually about fractions, (3) children benefit from opportunities to talk to one another and with their teacher about fraction ideas as they construct their own understandings of fraction as a number, and (4) teaching materials for fractions should focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer et al. 1997).

A decade of research on the teaching and learning of fractions among fourth and fifth graders has shown us that of the four pedagogical beliefs listed above, the second is the most important. In order to develop fraction sense, most children need extended periods of time with physical models such as fraction circles, Cuisenaire rods, paper folding, and chips. These models allow students to develop mental images for fractions, and these mental images enable students to understand about fraction size. Students can use their understanding of fraction size to operate on fractions in a meaningful way. The multiple models mentioned above were used in RNP teaching experiments. The fraction circle model used in combination with the RNP activities (see fig. 5.1)

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was the most powerful of the models. During interviews, students consistently referred to fraction circles as the model that helped them order fractions and estimate the reasonableness of fraction operations.

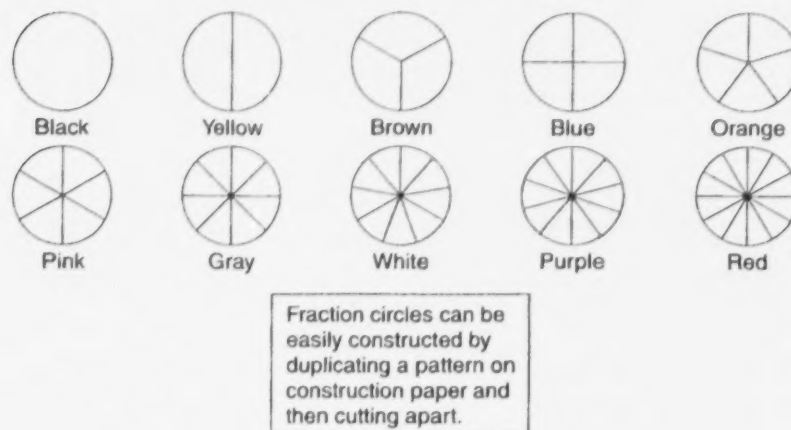


Fig. 5.1. Fraction circles

We know a traditional approach to fraction instruction does not foster students' number sense. Consider the following two fourth-grade students. Both were tracked into the middle mathematics group in a suburban school district south of Minneapolis. They were interviewed the same day as a part of the district's project to field-test the RNP curriculum. The interview was the final one given to both students at the end of five weeks of instruction. Jeremy was in a class using the RNP curriculum, whereas Annie used a traditional textbook program. Jeremy had not learned any procedures for ordering fractions but had interacted daily with fraction manipulatives, exploring fraction concepts and ideas involving ordering at the concrete level. Annie had few opportunities with manipulative materials, since the traditional program moved quickly into procedural skills dealing with fractions. Both children were asked to select the smaller of this pair of fractions: $\frac{4}{35}$ and $\frac{4}{29}$. Jeremy responded by reasoning, " $\frac{4}{29}$ is greater. Pieces are bigger, and there's 4 of the smaller pieces and 4 of the bigger piece." Annie reasoned, " $\frac{4}{29}$ is less. Well, nothing times 29 equals 35, and it's lower numbers."

Annie demonstrated what the RNP documented many times: When students do not have mental images for fractions, they resort to whole-number strategies to try and make sense of the problem (Post et al. 1985). In this instance, Annie's learned procedure did not help her with this problem, since one denominator was not a factor of the other denominator; she therefore

used a whole-number ordering strategy. Jeremy, by contrast, was able to reason about the fraction pair. Even though he did not have a concrete model to show 35ths or 29ths, he was able to transfer his thinking about ordering fractions with same numerators—which he did experience concretely in the RNP lessons—to a new situation. We can see that his language is tied to a manipulative model when he states that the “pieces are bigger.” During the lessons he used circular pieces of different colors as the main model. Jeremy’s thinking is typical of students who have used the RNP curriculum. Table 5.1 shows how he used mental images of fractions in three other ordering tasks from the same interview. Annie’s responses to the same questions are also shown. Notice Jeremy’s reliance on his mental pictures of the fraction circles.

TABLE 5.1
Other Examples of Students’ Ordering Strategies

Problem Task	Jeremy [RNP]	Annie [TEXT]
Order smallest to largest: $1/5$, $1/3$, and $1/4$	$1/5$, $1/4$, $1/3$. Pieces for $1/5$ are the smallest; for $1/4$ there’s four to cover the circle; $1/3$ there’s three.	$1/3$, $1/4$, $1/5$. I was thinking of the lowest numbers since all these [numerators] are the same.
$5/12$ and $3/4$: Are they equal or is one less?	$3/4$ is greater than $5/12$. $5/12$ ’s are red and not as big. It takes 3 reds to cover a fourth and there’s only five of them so it wouldn’t cover two of them.	$5/12$ is less. If you took 4×3 to get 12, common denominators and $3 \times 3 = 9$, it would be $9/12$ and it would be more.
$4/5$ and $9/10$: Are they equal or is one less?	$9/10$ is greater. The pieces are smaller because it takes 10, but there’s a smaller gap.	$4/5$ is less. $5 \times 2 = 10$ and $4 \times 2 = 8$ which gives $10/8$ and 8 is less than 9.

One purpose of this article is to share with teachers other examples of students’ thinking to give them a picture of what it means for students to exhibit number sense with fractions. Interviews with fourth graders who used the RNP curriculum and with fourth-grade students who used a traditional curriculum were conducted by RNP staff as well as classroom teachers. By looking at examples of conceptual and procedural thinking, teachers can construct for themselves a clearer picture of what they will want to observe in their own students. Focusing on children’s thinking and on the effect the use of manipulative models has on students’ thinking is a way teachers can learn to offer fraction lessons that involve their students with the use of manipulative models over an extended period of time.

ADDITION OF FRACTIONS

A child having fraction sense should be able to estimate a reasonable answer to fraction addition problems. Karen is an example of a fourth grader who was taught the RNP curriculum. She was asked to consider this problem on the day before she worked on fraction addition in her mathematics class (Cramer et al. 1997):

Jon calculated a problem as follows: $2/3 + 1/4 = 3/7$. Do you agree?

Karen: I don't agree. He did it weird. You don't add the top and bottom numbers.

Teacher: What would be an estimate?

Karen: It would be... greater than $1/2$ because $2/3$ is greater than $1/2$.

Teacher: Would it be greater or less than 1?

Karen: Less than 1. You'd need $1/3$, and $1/4$ is less than $1/3$.

Teacher: What about $3/7$?

Karen: $3/7$ is less than $1/2$.

Teacher: How do you know?

Karen: Because $3/7$ isn't $1/2$. I just know.

Notice how Karen, in thinking through this problem, used her knowledge of the relative size of fractions to bring meaning to the problem. She knew $2/3$ was greater than $1/2$. She did not do any symbolic procedure to compare fractions. She may have known from her many experiences with models for fractions that $2/3$ is greater than $1/2$. She also ordered $1/3$ and $1/4$. She realized that $2/3 + 1/3$ would equal 1, and since $1/4$ is less than $1/3$, the total would have to be less than 1. She was less clear about why $3/7$ was less than $1/2$, but she was sure that was true.

The RNP has documented how children who have had extended periods of time using manipulative models for fractions—and in particular, a circular model—construct for themselves informal ordering strategies to compare fractions (Post et al. 1985). We are using the term *informal strategies* to contrast with traditional ordering procedures of finding decimal equivalents or changing both fractions to equivalent fractions with like denominators. The traditional ordering algorithm is not as helpful in estimation problems as students' constructed strategies.

Karen used two different ordering strategies that the RNP has documented and which reflect students' use of mental images to judge the relative size of fractions (Bezuk and Cramer 1989). She used the same-numerator and the transitive strategies. In the earlier example, Jeremy demonstrated the

same-numerator strategy, which involves coordinating an inverse relationship between the size of the denominator and the size of the fraction. In concrete terms, students realize that the more equal parts a unit is partitioned into, the smaller each part is. Jeremy noted that since the numerator was 4 in both fractions, then four of the bigger parts would be larger than four of the smaller parts. Jeremy's thinking noted the role of both parts of the fraction, numerator and denominator. This is important, since often children who learn this relationship as an abstract rule inappropriately apply this thinking when the numerators are different. Karen also used this strategy to conclude that the sum of $2/3$ and $1/4$ was less than 1 because $1/4$ is less than $1/3$.

The transitive strategy involves using a reference point such as $1/2$. For example, Karen in another interview compared $5/12$ and $3/4$. She tried to apply an algorithm she had learned along the way to the problem, but she had difficulty. When asked to just think about the fractions in her mind, she responded, " $5/12$ is less than $1/2$; it's one less; $3/4$ is more toward the whole than $1/2$." What she meant by "one less" was that it was one red piece of a fraction circle away from $6/12$, which is $1/2$ (twelve reds equaled one circle). Karen used the transitive strategy when estimating fractions when she said that the sum had to be greater than $1/2$, since one of the addends, $2/3$, was already greater than $1/2$.

The use of the transitive and the same-numerator strategies was common among RNP students. Table 5.2 compares RNP and textbook students' estimation strategies (or lack of strategies) for a fraction addition problem. Students were randomly selected to participate in interviews given by project staff during the five instructional weeks. The problem in table 5.2 was on the final interview.

TABLE 5.2
Examples of Students' Fraction Estimation Strategies for the Number Line Problem

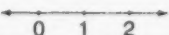
Problem	
Tell me about where $2/3 + 1/6$ would be on this number line: 	
Responses	
RNP Students	Textbook Students
Two browns are more than a half ... $1/6$ is not much more ... less one.	$2/3$ equals $4/6$... $4/6$ is less than $1/2$, so $2/3 + 1/6$ is less than $1/2$.
Between $1/2$ and 1, closer to 1 ... $2/3$ is almost one; $1/6$ is not equal to $1/3$, it is less than $1/3$.	I have no idea.

Table 5.2 (cont.)

RNP Students	Textbook Students
Between $1/2$ and 1 ... $2/3$ is a little more than $1/2$... $2/6$ equals $1/3$.	Student wouldn't estimate. He tried to find the exact answer. He was incorrect and used that as his estimate.
$2/3$ is greater than $1/2$, $1/6$ is a little more ... you need $1/6$ more to make a real whole.	Between $1\ 1/2$ and 2 ... just a guess.
Between $1/2$ and 1 ... $2/3$ is more than $1/2$; then add $1/6$. It's not greater than 1 because $1/6$ is less than $1/3$.	Maybe one ... I don't know.
Not a whole ... there's $2/3$, and $2/6$ is not enough to make $1/3$.	Between $1/2$ and 1 , closer to 1 ; just a guess.
$2/6$ fits over $1/3$, and there are 2 of them, and you add $1/6$ more ... $5/6$.	Student wouldn't estimate. He found the exact answer correctly and used that as his estimate.

Despite the same amount of instructional time and an emphasis on teaching the operations, textbook students did not have an adequate understanding of fraction size to find a reasonable estimate to a simple problem. RNP students' thinking depended on mental images for fractions and was directly related to their use of fraction circles. RNP students were better able to verbalize their thoughts. But then, the RNP lessons emphasized students' discussions (belief 3). In the lessons the manipulative model became the focal point of the discussion; students talked about their actions with the fraction circles. The extended use of manipulatives fostered students' verbal skills.

When estimating the sum of $2/3$ and $1/6$, RNP students reflected on how far $2/3$ is from 1 . Often RNP students considered the residual, or "leftover part," when they added fractions or when they ordered particular fraction pairs. The power the fraction circle model has on students' images for fractions is particularly evident in this type of reasoning. Consider Jeremy's use of residuals when he ordered $4/5$ and $9/10$ (see table 5.1). He imagined fraction circles and saw that both fractions had one part missing to make a whole. Since $1/10$ is smaller than $1/5$, less is missing for $9/10$; thus it is the bigger fraction. From this reasoning he knew that $9/10$ was greater than $4/5$. This type of reasoning required Jeremy to work through several steps mentally. His strong mental images were developed through extensive experiences with fraction circles and supported this multistep thought process.

Since the first RNP teaching experiment, we have documented this type of thinking among fourth graders using fraction circles (Cramer, Post, del Mas

in press). As part of the field testing of the RNP curriculum, each classroom teacher interviewed two or three students at the end of the five weeks of instruction. One of the questions was to order $\frac{4}{5}$ and $\frac{11}{12}$. An example of a residual strategy for this problem is shown by an RNP student. She stated that $\frac{4}{5}$ was less than $\frac{11}{12}$. Her reasoning was as follows: " $\frac{4}{5}$ has bigger pieces, but one piece is left; $\frac{11}{12}$ has smaller pieces, but only one piece is left. Fifth piece that is missing is bigger." The RNP curriculum did not directly teach students to order fractions by looking at the residual part, nor did it teach students to find common denominators. Results showed that 63 percent of the 53 RNP students interviewed ordered the pair correctly; 21 out of the 53 RNP students constructed for themselves an ordering strategy that relied on residuals. Only 3 of the 53 RNP students used a standard algorithm for ordering these fractions. A few correct solutions relied on drawings. Of the 57 textbook students interviewed, 67 percent ordered the fraction pair correctly. Only 9 out of 57 textbook students used a residual strategy to order these fractions; 18 out of 57 used a cross-product or common denominator method. A few used pictures and found the correct answer but could not explain their reasoning.

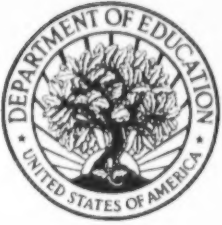
CONCLUSION

Often you hear teachers argue that there is not enough time to use manipulative materials. Even when manipulatives are used, teachers often make the transition to symbols too soon. The RNP students discussed in this article used manipulative models virtually every day during five weeks of instruction. The predominant model used was fraction circles. The samples of students' thinking presented here show the benefits of using manipulative models for five instructional weeks. RNP students developed number sense for fractions. In general, they had an understanding of fraction size evidenced by the type of ordering strategies they comfortably used. They were able to estimate reasonable answers to fraction addition problems. They were also able to verbalize their thinking. Students using a traditional program did not develop number sense.

Developing an understanding of fraction size and estimating a reasonable answer to fraction operation problems are appropriate goals for elementary school-aged children. Much of the symbolic manipulation of fraction symbols done in fourth and fifth grades can be adequately addressed in the middle grades. Students will be more successful if teachers in elementary school invest their time building meaning for fractions using concrete models and emphasizing concepts, informal ordering strategies, and estimation.

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